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ON CALCULATING TRANSIENT ELECTROMAGNETIC FIELDS OF A SMALL CURRENT-CARRYING LOOP OVER A HOMOGENEOUS EARTH

James R. Wait and Randolph H. Ott
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Boulder, Colorado 80302

Contract No. PRO-Y-71-872

Project No. 5635
Task No. 563506
Work Unit No. 56350601

Scientific Report No. 53
July 14, 1971

Contract Monitor: Philipp Blacksmith
Microwave Physics Laboratory

Prepared for
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
AIR FORCE SYSTEMS COMMAND
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1. INTRODUCTION

Remote sensing of the earth's surface layers often utilizes the eddy currents induced by an active electromagnetic source such as a magnetic dipole in an aircraft. The secondary fields are detected either at the aircraft or in a towed bird. While the main objective in commercial surveys is to locate metallic mineralization, the secondary field response is a measure of the effective ground conductivity of the overburden. This in turn can often be related to the rock strength. Currently, there is a great interest in utilizing transient source signals rather than the more prosaic time harmonic fields.

In this note, we wish to present the basic theory for the time domain response of a magnetic dipole or small loop located over an idealized flat homogeneous earth. The results are basic to a proper understanding of airborne electromagnetic geophysical surveys.

2. FORMULATION

Our model is very simple; we imagine the source to be a small loop (or vertical magnetic dipole) of area dA located at height h over a non-magnetic conducting half-space of conductivity σ. We choose a cylindrical coordinate system (ρ, ϕ, z) with the dipole at $z = h$ and the earth's surface at $z = 0$. We assume for sake of simplicity that all displacement currents may be neglected. This is justified by the fact that all linear dimensions of the problem are extremely small compared with the significant free space wavelengths in the problem.

The time harmonic solution for a current $I \exp(i\omega t)$ in the loop is well-known. The resulting magnetic field components at P are

$$H_\rho(i\omega) = \frac{\partial^2 \Pi(i\omega)}{\partial\rho\,\partial z} \tag{1}$$

and

$$H_z(i\omega) = \frac{\partial^2 \Pi(i\omega)}{\partial z^2} \tag{2}$$

where $\Pi(i\omega)$ is the Hertz potential given by

$$\Pi(i\omega) = \frac{I(i\omega)\,dA}{4\pi}\left[\frac{1}{r_o} - \frac{1}{r}\right]$$

$$+ \frac{I(i\omega)\,dA}{4\pi}\int_o^\infty \frac{2\lambda\, e^{-\lambda(h+z)}}{(\lambda^2 + \sigma\mu i\omega)^{\frac{1}{2}} + \lambda}\, J_o(\lambda\rho)\, d\lambda \tag{3}$$

where $r_o = [\rho^2 + (z-h)^2]^{\frac{1}{2}}$, $r = [\rho^2 + (z+h)^2]^{\frac{1}{2}}$, $\mu = 4\pi \times 10^{-7}$ and J_o is a Bessel function. If now the current in the loop is $I_o\,\delta(t)$, where $\delta(t)$ is the unit impulse function, we find without difficulty that the transient field components are to be obtained from

$$h_\rho(t) = \partial^2 \Pi(t)/\partial\rho\,\partial z \tag{4}$$

$$h_z(t) = \partial^2 \Pi(t)/\partial z^2 \tag{5}$$

where

$$\Pi(t) = \frac{I_o\,dA}{4\pi}\left[\frac{1}{r_o} - \frac{1}{r}\right]\delta(t) + \Delta\Pi(t) \tag{6}$$

and

$$\Delta\Pi(t) = \frac{I_o\,dA}{4\pi}\int_o^\infty \left[\mathcal{L}^{-1}\frac{2\lambda}{(\lambda^2 + \sigma\mu s)^{\frac{1}{2}} + \lambda}\right] e^{-\lambda(h+z)}\, J_o(\lambda\rho)\, d\lambda \tag{7}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform operator.

3. THE TRANSIENT SOLUTION

Using Pair No. 543.5 in Campbell and Foster's [1949] tables, we find that (7) is equivalent to

$$\Delta\Pi(t) = \frac{I_o\,dA}{4\pi}\,\frac{2}{\sqrt{\pi}}\,\frac{1}{t}\int_o^\infty F\left(\frac{\lambda^2 t}{\sigma\mu}\right) J_o(\lambda\rho)\, e^{-\lambda(h+z)}\, d\lambda \tag{8}$$

where

$$F(T) = T^{\frac{1}{2}} e^{-T}\left[1 - (\pi T)^{\frac{1}{2}} e^{T} \operatorname{erfc}(T^{\frac{1}{2}})\right] \tag{9}$$

and where erfc is the complement of the error function. By changing the variable of integration in (8) to $x = [t/(\sigma\mu)]^{\frac{1}{2}}\lambda$, we can write

$$\Delta\Pi(t) = \frac{I_o\,dA}{4\pi}\,\frac{2}{\sqrt{\pi}}\,\frac{1}{t}\left(\frac{\sigma\mu}{t}\right)^{\frac{1}{2}}\int_o^\infty \left[x e^{-x^2} - \pi^{\frac{1}{2}} x^2 \operatorname{erfc}(x)\right] e^{-Hx} J_o(Px)\, dx \tag{10}$$

where $H = (\sigma\mu/t)^{\frac{1}{2}}(h+z)$ and $P = (\sigma\mu/t)^{\frac{1}{2}}\rho$. In order to deal effectively with the integral in (10), we write

$$e^{-Hx} J_o(Px) = \sum_{m=o}^{\infty} a_m \frac{(-1)^m}{m!}\, x^m \tag{11}$$

where $a_m = (-1)^m (\partial^m/\partial x^m)\left[e^{-Hx} J_o(Px)\right]\Big|_{x=o}$. In the special case where $P = 0$ (i.e., $\rho = 0$), we see that $a_m = H^m$.

Using (11), we find that (10) becomes

$$\Delta\Pi(t) = \frac{I_o\,dA}{4\pi}\,\frac{2}{\sqrt{\pi}}\,\frac{1}{t}\left(\frac{\sigma\mu}{t}\right)^{\frac{1}{2}}\sum_{m=o}^{\infty}\frac{(-1)^m a_m}{m!}\left\{\int_o^\infty x^{m+1} e^{-x^2}\, dx \right.$$

$$\left. -\sqrt{\pi}\int_o^\infty x^{m+2} \operatorname{erfc}(x)\, dx\right\} \tag{12}$$

Now it follows from the basic definition of the Gamma function $\Gamma(n)$ of argument n that

$$\int_o^\infty x^{2n-1} e^{-x^2}\, dx = \Gamma(n)/2 \tag{13}$$

which has the form of the first integral in (12). Also, from Gradshteyn and Ryzhik [1965], (No. 6281, pg. 648), we have

$$\int_{o}^{\infty} x^{2q-1} \operatorname{erfc}(x)\, dx = \frac{\Gamma(q+\frac{1}{2})}{2q\pi^{\frac{1}{2}}} \tag{14}$$

which is the form of the second integral in (12). Thus

$$\Delta \Pi(t) = \frac{I_o dA}{4\pi} \frac{1}{t} \left(\frac{\sigma\mu}{\pi t}\right)^{\frac{1}{2}} \sum_{m=o}^{\infty} \frac{(-1)^m a^m}{m!} \frac{\Gamma\left(\frac{m}{2}+1\right)}{m+3} \tag{15}$$

In order to obtain explicit series representations for the field components, we need to differentiate (15) according to (4) and (5). Of special interest is the response $h_z(t)$ in the case $\rho = 0$. Then we find without difficulty that

$$h_z(t) = \frac{I_o dA}{4\pi} \frac{\partial^2}{\partial z^2} \left(\frac{1}{r_o} - \frac{1}{r}\right) + \Delta h_z(t) \tag{16}$$

where

$$\Delta h_z(t) = \frac{\partial^2}{\partial z^2} \Delta \Pi(t)$$

$$= \frac{I_o dA}{4\pi} \frac{1}{\sqrt{\pi}} \frac{1}{t} \left(\frac{\sigma\mu}{t}\right)^{3/2} \sum_{m=2}^{\infty} \frac{(-1)^m m(m-1) H^{m-2} \Gamma\left(\frac{m}{2}+1\right)}{m!\,(m+3)} \tag{17}$$

where $H = (\sigma\mu/t)^{\frac{1}{2}}(h+z)$. From (17), it is evident that for large values of t

$$\Delta h_z(t) = \frac{I_o dA}{10\pi^{3/2}} \frac{1}{t} \left(\frac{\sigma\mu}{t}\right)^{3/2} \tag{18}$$

which indicates that the magnetic field response varies as $t^{-5/2}$ for sufficiently large times. This result is in agreement with a similar recent analysis of the problem by Ott [1971], but it disagrees with the analysis by Keller, et al [1970], who made unjustified approximations in the integrand of the time-harmonic

integral representation. It is remarkable that (18) does not depend explicitly on the distance between source and observer. This, of course, would not be the case in general. Nevertheless, it suggests that for a vertical co-axial loop system, the transient coupling should be relatively insensitive to height of the device above the ground.

The behavior of the transient response at small times t can be obtained directly from (8) and (9). For example, if t is sufficiently small $F(T) \simeq T^{\frac{1}{2}}$ and then

$$\Delta\Pi(t) \simeq \frac{I_o dA}{4\pi} \cdot \frac{2}{\sqrt{\pi}} \frac{1}{t}\left(\frac{t}{\sigma\mu}\right)^{\frac{1}{2}} \int_0^\infty \lambda J_o(\lambda\rho)\, e^{-\lambda(h+z)}\, d\lambda$$

$$\simeq \frac{I_o dA}{2\pi}\left(\frac{1}{\pi\sigma\mu t}\right)^{\frac{1}{2}} \frac{\partial}{\partial z}\frac{1}{r} \qquad (19)$$

This shows that $\Delta\Pi(t)$ and the field components $h_\rho(t)$ and $h_z(t)$ vary as $t^{-\frac{1}{2}}$ for sufficiently small times.

In the general case where time t is neither small nor large, it appears necessary to evaluate numerically the integral of the type given by (10). Some examples from this approach are given in Figure 1 where we plot $h_z(t)$ and $h_\rho(t)$ normalized to $4\pi\sigma\mu\rho^5/I_o dA$ as a function of the time parameter $T = t/\sigma\mu\rho^2$ for various values of the ratio, $R = (h+z)/\rho$. The ordinates are labelled $H_z(T)$ and $H_\rho(T)$ to denote the normalization.

The small time and the large time asymptotes are also shown on the curves. These are discussed in the next section.

4. DISCUSSION OF RESULTS AND CONCLUDING REMARKS

The first two leading terms in the large time response are found to be

$$H_z(T) \sim \frac{1}{5\sqrt{\pi}T^{5/2}}\left(1 - \frac{5\sqrt{\pi}R}{8\sqrt{T}}\right) + O\left(\frac{1}{T^{7/2}}\right),\ t \sim +\infty \qquad (20)$$

This result is compared with the numerical results in Figure 1 for a height to separation ratio, R, of 1/2. The results of (20) and the numerically computed results show very good agreement for large times.

From (8) and (9), the small time response, for $t \sim +0$ is given by

$$H_z(T) \sim \frac{6R}{\sqrt{\pi}\,T} \frac{(2R^2 - 3)}{(1+R^2)^{7/2}} \tag{21}$$

This small time response is compared with the numerical results in Figure 1 for $R = 1/2$. Again, good agreement between the methods is obtained. An interesting feature of the small time response in (21) may be noted by varying R for fixed T. As we vary R, there should be a minimum in the small time response at $R = \sqrt{3}/\sqrt{2}$. Indeed, as we vary R in Figure 1, we find that increasing R to $3/2$ increases the response, and decreasing R to 1 also increases the response.

As indicated in Figure 1, we find that the numerical results for $h_z(t)$ show a reversal in polarity, while those for $h_\rho(t)$ do not. This change in polarity does not occur for $h_z(t)$ for values of R greater than about 1.30. Obviously, from Figure 1, the small and large time responses are not sufficient to completely describe the response at intermediate times. This points up the necessity of dealing with the full integral representation of the problem if a proper understanding of the transient phenomena is to be achieved.

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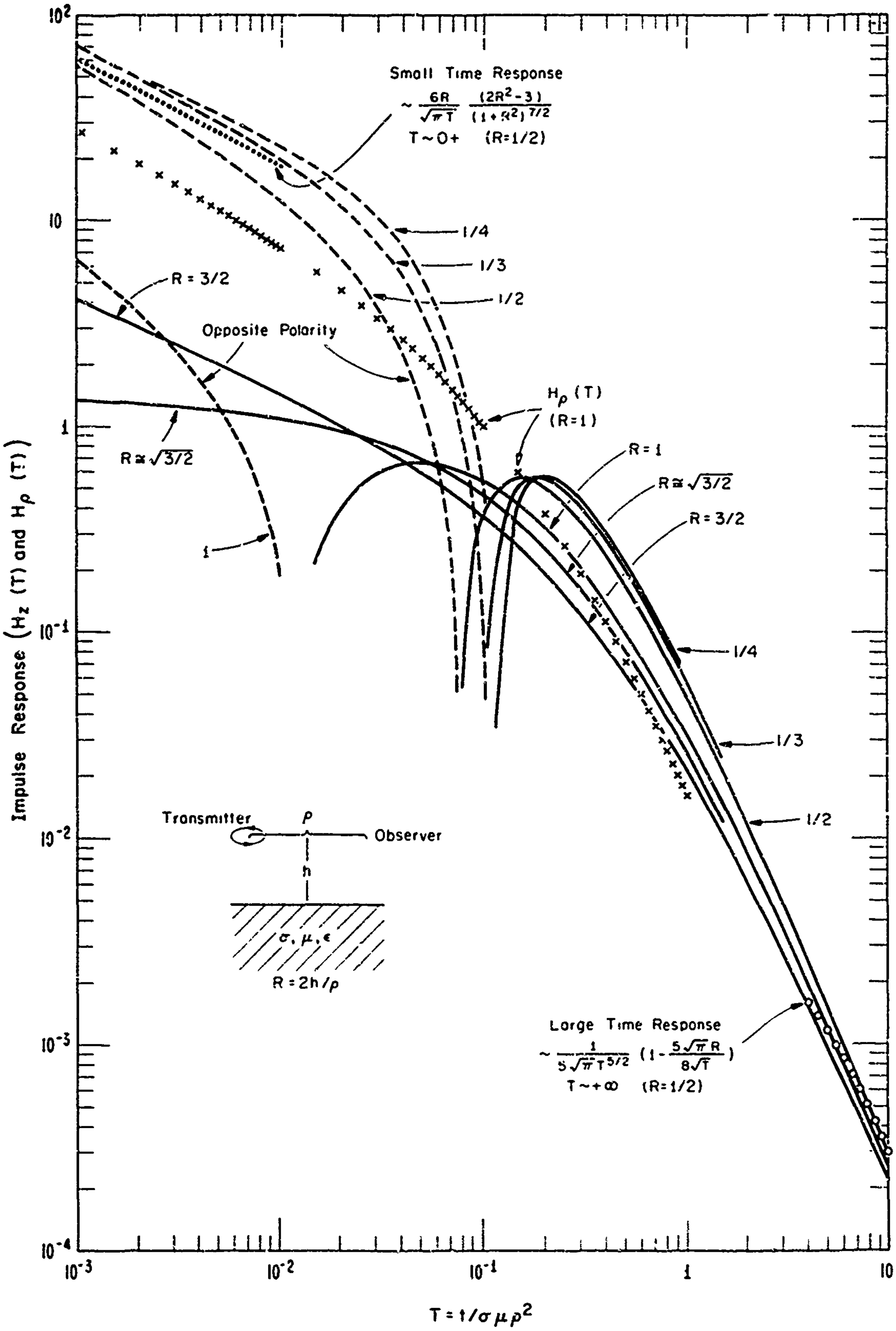

Fig. 1 Impulse field responses $H_z(T)$ as a function of normalized time for various R values. $H_\rho(T)$ is shown for R = 1 only.